NONSTATIONARY GAS FLOW WITH SHOCK WAVES IN A SUPERSONIC COMPRESSOR

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Shock waves are formed in the channels between blades in a compressor working in the transsonic state, and the positions of these vary periodically and produce strong vibrations in the blades. The effect is extremely complex and is dependent on a large number of parameters. Here we present a simplified model for the effect, which can be examined theoretically. It is assumed that the nonstationary pulsations in the flow and the amplitudes in the oscillations of the shock waves are small, and therefore one can employ a steady-state flow whose characteristics may be taken as given, including the mean position of the shock waves.

1. We consider a planar potential flow of barotropic gas in a set of thin slightly curved blades; the flow at the inlet is assumed unperturbed and supersonic, while at the outlet it is subsonic. Within each channel between the blades there is a straight shock wave of small intensity; the flow beyond the shock wave then remains potential to a first approximation. The shapes of the blades oscillate in accordance with a specified harmonic law having a circular frequency ω and a constant phase shift μ between adjacent blades. The amplitude of the oscillation is considered as small relative to the chord b of the blades. In general, one gets behind the blades a system of eddies due to the change in the circulation around each blade. The eddies can be simulated as lines of contact discontinuity, which are considered to be disposed along straight lines that constitute continuations of the blade cores.

To each blade we assign in sequence the numbers $n=0, \pm 1, \ldots$ (Fig. 1); the first blade ($n=0$) is linked to a cartesian coordinate system x, y with its origin at the leading edge of the nonvibrating blade. The x axis lies along the chord of that blade, while the y axis is directed upwards. Let β be the angle of attack, h the pitch of the blades, and $L_n^{(1)}$, $L_n^{(2)}$ the contours of the upper and lower sides of blade n respectively, with Ω_n flow region bounded by the front of the set of blades, the contours $L_n^{(1)}$, $L_{n+1}^{(2)}$ and the lines of contact discontinuity (Fig. 1):

$$C_n : x \geqslant b + nh \sin \beta, \quad y = nh \cos \beta$$
$$C_{n+1} : x \geqslant b + (n+1) h \sin \beta, \quad y = (n+1) h \cos \beta \tag{1.1}$$

The velocity potential φ (x, y, t) satisfies the following equation in the general case for the region Ω_n and D_n:

$$[a_1^2 - (\varkappa - 1)\{\varphi_t + {}^1/_2[(\nabla\varphi)^2 - \mathbf{V}_1\cdot\mathbf{V}_1]\}]\,\Delta\varphi = \varphi_{tt} + \left[\frac{\partial}{\partial t} + \frac{1}{2}(\nabla\varphi\cdot\nabla)\right](\nabla\varphi)^2 \tag{1.2}$$

where $\mathbf{V}_1$ and a_1 are, respectively, the velocity vector and the speed of sound in the unperturbed supersonic flow in front of the blades, while $\varkappa$ is the adiabatic constant and t is time.

The following are the boundary conditions in the regions Ω_n, in which the gas flow is supersonic: the condition that the gas does not flow through the contour $L_n = L_n^{(1)} + L_n^{(2)}$ ($n=0, \pm 1, \ldots$)

$$\{\nabla\varphi - \mathbf{V}_1 - \mathbf{w}_t^{(n)}\}\cdot\boldsymbol{\nu}_n = 0 \quad \text{for} \quad (x, y) \in L_n \tag{1.3}$$

and the absence of perturbations in the gas along the Mach lines diverging from the leading edges:

$$\varphi\,(x,\ y,\ t) = \mathbf{V}_1\cdot\mathbf{r} \tag{1.4}$$

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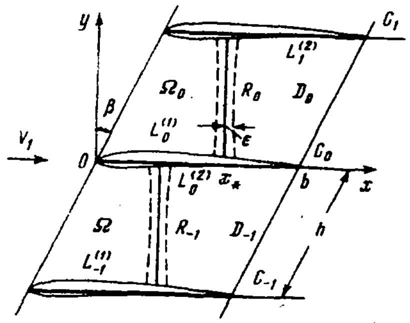

Fig. 1

Here $\mathbf{r}$ is the radius vector of the point, ν_n is the normal to L_n, and $\mathbf{w}^{(n)}$ is the vector for the displacement of the points on contour L_n.

The flow is subsonic in regions D_n; the boundary conditions for these regions are: condition (1.3) above at the points on L_n, and the condition that the normal component of the gas velocity behind R_n is equal to some value φ_x^+, which will be derived below:

$$\varphi_x = \varphi_x^+ \quad \text{for } (x, y) \in R_n \quad (n = 0, \pm 1, \ldots) \tag{1.5}$$

together with the condition of continuity for the pressure p on the lines of contact discontinuity:

$$[p] = 0 \quad \text{for } (x, y) \in C_n \quad (n = 0, \pm 1, \ldots) \tag{1.6}$$

We also assume that Zhukovskii's postulate about the pressure continuity applies to the rear edges of all blades.

The initial conditions are not specified as we are assuming a harmonic law of motion for the gas as a function of time.

Condition (1.5) contains the unknown function φ_x^+, which has to be determined from equations for the R_n ($n = 0, \pm 1, \ldots$):

$$\begin{gathered}\rho_- \theta_- (\varphi_x^+ - \varphi_x^-) = p_+ - p_-, \quad \rho_- \theta_- = \rho_+ \theta_+ \\ p_- [(\varkappa + 1)\rho_- - (\varkappa - 1)\rho_+] = p_- [(\varkappa + 1)\rho_+ - (\varkappa - 1)\rho_-]\end{gathered} \tag{1.7}$$

Here ρ is gas density and θ is the propagation speed of the shock wave, while the subscripts minus and plus define the limiting values of the hydrodynamic quantities as one approaches R_n from left and right, respectively, while the pressure p is related to the velocity potential φ by the Cauchy–Lagrange integral

$$\int_{p_1}^{p} \frac{dp}{\rho} + \frac{\partial \varphi}{\partial t} + \frac{1}{2}[(\nabla\varphi)^2 - V_1^2] = 0 \tag{1.8}$$

where p_1 is the pressure in the unperturbed flow.

System (1.7) contains 7 unknown functions: the displacement speed N of the shock wave, the velocities φ_x^+, φ_x^-, the densities ρ_+, ρ_- and also φ_t^+, φ_t^-; of these, only three quantities (φ_x^-, φ^-_t, and ρ_-) are determined by solving (1.2)-(1.4) in the supersonic region, so (1.7) forms an unclosed system.

To obtain the additional equation we use the condition for continuity in the tangential components of the the velocities $\varphi_y^+ = \varphi_y^-$ at the straight shock wave, which gives

$$[\varphi] = \text{const} \quad \text{for } (x, y) \in R_n \quad (n = 0, \pm 1, \ldots) \tag{1.9}$$

We differentiate (1.9) with respect to t and get a further condition for R_n:

$$\varphi_t^+ = \varphi_t^- \tag{1.10}$$

which closes system (1.7).

2. We represent the velocity potential $\varphi(x, y, t)$ in the form

$$\varphi(x, y, t) = \varphi_0(x, y) + ba_1 \Phi(x, y) e^{i\omega t} \tag{2.1}$$

where $\varphi_0(x, y)$ is that potential for the steady-state flow of the gas around the motionless blades, and $\Phi(x, y)$ is a dimensionless amplitude function for the velocity potential of the additional nonstationary flow.

We assume that the velocity distribution for the stationary flow in regions Ω_n and D_n is not dependent on $n = 0, \pm 1, \ldots$ and differs little from the velocity distribution for the corresponding averaged motion. This assumption goes with the smallness of the vibration amplitude to give us linear equations for $\Phi(x, y)$ in the regions Ω_n and D_n, these being independent of the parameters of the stationary motion. The dependence of Φ on these parameters appears only via condition (1.7) at the shock waves. Here, as will be shown, it is sufficient to know the mean position of the shock wave in the channel between blades in order to calculate the nonstationary flow, provided that the magnitude of the pressure discontinuity in the stationary flow at the shock wave is known. We assume in what follows that these parameters are given.

Then within the framework of these assumptions it is sufficient to consider the solution for $\Phi(x, y)$ in the linear formulation; we denote by Φ_1 and Φ_2 the values of Φ in the supersonic and subsonic flow regions respectively. Then Φ in the flow region should satisfy

$$(M_k^2 - 1)\Phi_{kxx} - \Phi_{kyy} + \frac{2i\omega}{a_k} M_k \Phi_{kx} - \frac{\omega^2}{a_k^2}\Phi_k = 0 \qquad \left(M_k = \frac{V_k}{a_k};\ k = 1, 2\right) \tag{2.2}$$

where V_2 and a_2 are certain average values of the flow velocity and speed of sound in the subsonic flow region.

The following are the boundary conditions for the regions Ω_n ($n = 0, \pm 1, \ldots$):

$$\Phi_{1y} = M_1 W_x^{(n)} + i\omega a_1^{-1} W^{(n)} \quad \text{for } x_n < x < x_* + x_n,\ y = y_n \tag{2.3}$$
$$(x_n = nh \sin\beta,\ y_n = nh\cos\beta)$$
$$\Phi_1 = 0 \tag{2.4}$$

on the Mach lines diverging from the leading edges.

The following are the boundary conditions for Φ_2 in the regions D_n ($n = 0, \pm 1, \ldots$):

$$\Phi_{2y} = \left[M_2 W_x^{(n)} + \frac{i\omega}{a_2} W^{(n)}\right]\frac{a_2}{a_1} \equiv F^{(n)}(x) \tag{2.5}$$
$$\text{for } x_* + x_n < x < b + x_n,\ y = y_n$$

$$\Phi_{2x} = \Phi_{2x}^{+} \quad \text{for } x = x_* + x_n,\ y_n < y < y_{n+1} \tag{2.6}$$
$$[i\omega\Phi_2 + V_2\Phi_{2x}] = 0 \quad \text{for } x \geqslant b + x_n,\ y = y_n \tag{2.7}$$

Here the boundary conditions have been transferred to straight lines parallel to the x and y axes, while x_* is the abscissa of the mean position of the shock wave at the upper side of the initial blade, while the dimensionless function $W^{(n)}(x)$ is defined by

$$w^{(n)}(x, t) = bW^{(n)}(x)\, e^{i\omega t} \tag{2.8}$$

where $w^{(n)}$ is the projection on the y axis for the displacement vector of blade L_n. Also, (2.7) combines the condition at the eddies and the Zhukovskii postulate.

The problem of (2.2) and (2.5)-(2.7) is related to that of (2.2)-(2.4) by the equations for the discontinuity (1.7) and the additional condition

$$\Phi_1^{+} = \Phi_1^{-} \tag{2.9}$$

which is a consequence of (1.10).

3. The law chosen for the vibrations of the blades gives

$$W^{(n)}(x) = W^{(0)}(x)\, e^{in\mu}, \qquad F^{(n)}(x) = F^{(0)}(x)\, e^{in\mu} \tag{3.1}$$

Then as the boundary-value problem of (2.2)-(2.7) is linear, it follows that Φ satisfies the generalized periodicity condition

$$\Phi(x + nh\sin\beta,\ y + nh\cos\beta) = \Phi(x, y)\, e^{in\mu} \tag{3.2}$$

Condition (3.2) enables one to restrict the solution to the regions Ω_0 and D_0; the solution to (2.2)-(2.4) in region Ω_0 is not dependent on the conditions of (1.7) at R_0 and can be obtained by methods presented in [1, 2].

We now pass to construction of the solution for region D_0; we introduce the amplitude function $\Psi(x, y)$ for the acceleration potential, which is related to $\Phi_2(x, y)$ by

$$\Psi = i\omega\Phi_2 + V_2\Phi_{2x} \tag{3.3}$$

The inverse relationship of Φ_2 to Ψ is

$$\Phi_2(x, y) = \Phi_2(x_*, y) + \frac{1}{V_2} e^{-i\omega x/V_2} \int_{x_*}^{x} \Psi(u, y)\, e^{i\omega u/V_2}\, du \tag{3.4}$$

The definition of (3.3) and the condition of (2.7) together mean that Ψ is continuous everywhere in the region $D = \sum_{-\infty}^{\infty}(D_n + C_n)$ and satisfies (2.2) with k=2 in this region; therefore, it remains to meet conditions (2.5) and (2.6), which on the basis of (3.1) and (3.4) take the form

$$V_2 [F^{(0)}(x) - F^{(0)}(x_*)] = e^{-i\omega x/V_2} \lim_{y\to 0} \int_{x_*}^{x} \Psi_y(u, y) e^{i\omega u/V_2} du \quad (x_* - x_1 < x < b) \tag{3.5}$$

$$V_2 \Phi_{2x}^{+}(y) = \Psi(x_*, y) \quad (0 \leqslant y \leqslant y_1) \tag{3.6}$$

We seek Ψ by superposition of singularities, for which purpose we dispose along the straight line $x = x_* + x_n$, $y_n < y < y_{n+1}$ ($n = 0, \pm 1, \ldots$), while along straight lines $x_* + x_n - x_1 < x < b + x_n$, $y = y_n$ we place a system $q_n(y)$ of dipoles $\gamma_n(x)$; (3.2) implies that

$$q_n(y) = q(y) e^{in\mu}, \quad \gamma_n(y) = \gamma(y) e^{in\mu} \tag{3.7}$$

Then the Ψ that satisfies (2.2) can be represented for the points $(x, y) \in D_0$ and

$$\begin{aligned}\Psi(x, y) = \sum_{n=-\infty}^{\infty} e^{in\mu} \Big\{ \int_{y_n}^{y_{n+1}} q(\eta) H_0^{(2)}(r_{1n}) \times \\ \times \exp\left[\frac{i\omega M_2}{a_2(1 - M_2^2)}(y - \eta)\right] d\eta + \int_{x_*+x_n-x_1}^{b+x_n} \gamma(\xi) \times \\ \times \frac{y - nh\cos\beta}{r_{2n}} H_1^{(2)}(r_{2n}) \exp\left[\frac{i\omega M_2}{a_2(1 - M_2^2)}(x - \xi)\right] d\xi \Big\}\end{aligned} \tag{3.8}$$

where $H_0^{(2)}$, $H_1^{(2)}$ are Hankel functions of the second kind, while

$$\begin{aligned} r_{1n} &= \frac{\omega}{a_2(1 - M_2^2)} \sqrt{(x - x_* - x_n)^2 + (1 - M_2^2)(y - \eta)^2} \\ r_{2n} &= \frac{\omega}{a_2(1 - M)_2^2} \sqrt{(x - \xi)^2 + (1 - M_2^2)(y - y_n)^2} \end{aligned} \tag{3.9}$$

We substitute (3.8) into (3.5) and (3.6) to get a system of two integral equations for $q(\eta)$ and $\gamma(\xi)$. Here one must bear in mind that the second group of terms in (3.8) has a singularity at $\xi = x$.

4. Equation (3.6) contains the unknown function Φ_{2x}^{+}, which should be determined from (1.7) and (1.10); we consider this in more detail. The first of the equations in (1.7) contains the propagation speed for the straight shock wave R_0, which is defined by

$$\theta = N - \varphi_x \quad (N = \partial x_W/\partial t, \; x_W(y, t) = x_* + \mathrm{Re}\{\varepsilon(y)\, e^{i\omega t}\}) \tag{4.1}$$

Here N is the displacement speed of R_0 in the direction of the x axis, while x_W is the abscissa of the shock wave on $L_0^{(1)}$ at time t, and $\varepsilon(y)$ is the complex amplitude of the shock wave.

We represent the pressure and density in the form

$$p = p_0 + \mathrm{Re}\{p' e^{i\omega t}\}, \quad \rho = \rho_0 + \mathrm{Re}\{\rho' e^{i\omega t}\} \tag{4.2}$$

where p_0 and ρ_0 are the pressure and density in the stationary flow, while p' and ρ' are the complex amplitude functions for nonstationary components of the pressure and density, respectively.

We substitute (2.1), (4.1), and (4.2) into (1.7) and use (1.10) to get

$$\begin{aligned} &p_0^+ - p_0^- = \rho_0^-(V_1 + \varphi_{0x}^-)(\varphi_{0x}^- - \varphi_{0x}^+) \\ &\rho_0^+(V_1 + \varphi_{0x}^+) = \rho_0^-(V_1 + \varphi_{0x}^-) \\ &p_0^+[(\varkappa + 1)\rho_0^- - (\varkappa - 1)\rho_0^+] = p_0^-[(\varkappa + 1)\rho_0^+ - (\varkappa - 1)\rho_0^-] \end{aligned} \tag{4.3}$$

$$(p_+' - p_-')/\rho_0^- = (\varphi_{0x}^- - \varphi_{0x}^+)[ba_1\Phi_x^- - i\omega\varepsilon + (V_1 + \varphi_{0x}^-)\rho_-'/\rho_0^-] + ba_1(V_1 + \varphi_{0x}^-)(\Phi_x^- - \Phi_x^+) \tag{4.4}$$

$$ba_1[(V_1 + \varphi_{0x}^-)\Phi_x^+ - (V_1 + \varphi_{0x}^+)\Phi_x^-] = i\omega\varepsilon(\varphi_{0x}^- - \varphi_{0x}^+) + (V_1 + \varphi_{0x}^+)(V_0 + \varphi_{0x}^-)[\rho_-'/\rho_0^- - \rho_+'/\rho_0^+] \tag{4.5}$$

$$p'_+/p_0^+ - \varkappa\rho'_+/\rho_0^+ = 0, \quad p'_-/p_0^- - \varkappa\rho_-'/\rho_0^- = 0 \tag{4.6}$$

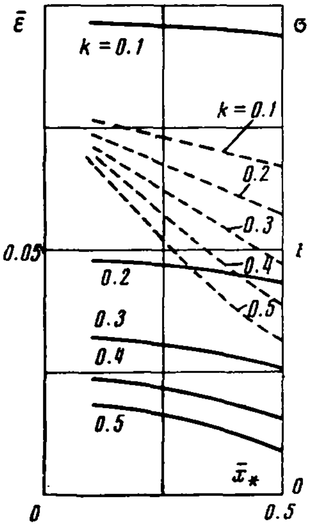

Fig. 2

Equations (4.3) relate the hydrodynamic quantities on the different sides of the shock waves in the stationary flow, while (4.4)-(4.6) establish an additional relationship for the nonstationary flow. The latter relationships are correct to quantities of the first order of smallness and enable us to determine Φ_x^+ and ε via the known hydrodynamic parameters of the supersonic flow. For this purpose it is convenient to use the accessory relationship

$$\frac{p'}{p_0} - \frac{\rho'}{\rho_0} = -\frac{\varkappa - 1}{\varkappa p_0} b a_1 \rho_0 \left[(V_1 + \varphi_{0x}) \Phi_x + i\omega\Phi \right] \tag{4.7}$$

which follows from the equality of the coefficient to $\Delta\varphi$ in (1.2) to the square of the speed of sound $a^2 = \varkappa p/\rho$; then (4.7) gives from (4.4)-(4.6) the following expressions for Φ_x^+, ε:

$$\Phi_x^+ = \frac{1}{(\varkappa - 1)(V_1 + \varphi_{0x}^-) - (\varkappa + 1)(V_1 + \varphi_{0x}^+)} \times$$
$$\times \left\{ [(\varkappa + 1)(V_1 + \varphi_{0x}^-) - (\varkappa - 1)(V_1 + \varphi_{0x}^+)] \Phi_x^- - \frac{i\omega(\varkappa - 1)}{\varkappa + 1} \frac{(\varphi_{0x}^- - \varphi_{0x}^+)^2}{(V_1 + \varphi_{0x}^-)^2} \Phi^- \right\} \tag{4.8}$$

$$\varepsilon = -\frac{iba_1(\varphi_{0x}^- - \varphi_{0x}^+)}{\omega[(\varkappa - 1)(V_1 + \varphi_{0x}^-) - (\varkappa + 1)(V_1 + \varphi_{0x}^+)]} \left\{ (\varkappa + 1)\Phi_x^- + \frac{i\omega(\varkappa - 1)}{V_1 + \varphi_{0x}^-} \Phi^- \right\} \tag{4.9}$$

Formulas (4.8) and (4.9) completely close the system of (2.2)-(2.7) for the additional velocity potential of the nonstationary motion.

5. We now determine the nonstationary components of the lift ΔP_n and torque ΔM_n acting on blade n; from (3.2) we have

$$\Delta P_n = \mathrm{Re}\{\Delta P e^{i(\omega t + n\mu)}\}, \qquad \Delta M_n = \mathrm{Re}\{\Delta M e^{i(\omega t + n\mu)}\} \tag{5.1}$$

where ΔP and ΔM are the complex amplitudes of the above quantities on the initial blade; the values of these are dependent on the pressure distribution along the vibrating blade and on the displacement amplitude of the shock wave on the upper and lower sides of the blade, together with the discontinuity in the stationary pressure at the shock waves. We have as follows for the results for the supersonic and subsonic flows:

$$\Delta P = \int_0^{x_* - x_1} p_1'(x, -0)\,dx + \int_{x_* - x_1}^{b} p_2'(x, -0)\,dx - \int_0^{x_*} p_1'(x, +0)\,dx -$$
$$- \int_{x_*}^{b} p_2'(x, +0)\,dx + (p_0^+ - p_0^-)[\varepsilon(0) - \varepsilon(y_1) e^{-i\mu}] \tag{5.2}$$

$$\Delta M = \int_0^{x_* - x_1} x p_1'(x, -0)\,dx + \int_{x_* - x_1}^{b} x p_2'(x, -0)\,dx -$$
$$- \int_0^{x_*} x p_1'(x, +0)\,dx - \int_{x_*}^{b} x p_2'(x, +0)\,dx + (p_0^+ - p_0^-)[x_* \varepsilon(0) - (x_* - x_1)\varepsilon(y_1) e^{-i\mu}] \tag{5.3}$$

Here p_1', p_2' are the values of p' in the supersonic and subsonic flow regions respectively, while the aerodynamic torque is calculated relative to the leading edge of a blade; (5.1)-(5.3) show that one needs to know Φ_1 and Φ_2 in order to calculate the nonstationary hydrodynamic reactions at the blades within the framework of this model, together with the oscillation amplitude of the shock waves, the mean position of the shock waves, and the discontinuity in the stationary pressure at these waves on the initial blade.

As an example, Fig. 2 shows results for the dimensionless vibration amplitude $\bar{\varepsilon} = |\varepsilon|/b$ (solid lines) and the phase σ (broken lines) for the shock wave ($\varepsilon = |\varepsilon| e^{i\sigma}$) for a set of blades in relation to the mean position of the shock wave $\bar{x}_* = x_*/b$ for various values of $k = \omega b/a_1$.

The blades perform antiphase torsional vibrations about the leading edge with amplitude $|\alpha| = 0.01$; the calculation has been performed for $M_1 = 1.1$, $\varphi_{0x}^- = 0.05$, $\varphi_{0x}^+ = -0.225\ a_1$.

The results show that the oscillation amplitude of the shock wave tends to diminish as one recedes from the leading edge, while there is a marked dependence of $\bar{\varepsilon}$ on the Strouhal number. Also, in this example σ varies in the range $[0, \pi]$, which means that the torsional oscillations are damped by the shock waves.

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